Some existing approaches

- **Multivariate moment matching approach** (Weile et al. 99, Daniel et al. 04)
  - Moment matching about the Laplace variable $s$ and the parameter $p$.
  - Affine parameter dependency is required
  - Curse of dimensionality (reduced order grows rapidly even for small numbers of parameters)

- **Common projection approach** (Leung et al. 05, Li et al. 05, Peng et al. 05)
  - Common projection matrix calculated from several local models
  - Moment matching property for each of the local models
  - Reduced order depends on the number of local models considered
  - Affine parameter dependency is required to obtain a parametric reduced model

- **TBR-Interpolation-based approach** (Baur et al. 08, 09)
  - Interpolation between TFs of locally reduced systems obtained by TBR
  - Benefits from error bounds and stability of TBRs.
  - Reduced order depends on the number of the local models considered
  - Lightly damped modes can cause problems

Can a new approach avoid some of the disadvantages?
Part 1:
Interpolation between locally reduced models as a \textit{framework} for parametric reduction

Starting Point
System:
\[ \dot{x} = A(p)x + b(p)u, \quad y = c^T(p)x \]

Matrices $A, b, c$ only available at discrete values $p_1, p_2, \ldots$ of $p$:
- $A(p_1) = A_1$, $A(p_2) = A_2$, \ldots
- $b(p_1) = b_1$, $b(p_2) = b_2$, \ldots
- $c(p_1) = c_1$, $c(p_2) = c_2$, \ldots
Interpolation of system coefficients

Linear Interpolation of coefficients (system matrices):

\[
\dot{x} = \left( \sum_{i=1}^{s} \omega_i(p) A_i \right) x + \left( \sum_{i=1}^{s} \omega_i(p) b_i \right) u, \quad y = \left( \sum_{i=1}^{s} \omega_i(p) c_i^T \right) x
\]

= exact description if \( p \) affine: \( A = A_0 + A_1 p \), \( b = b_0 + b_1 p \), \( c = c_0 + c_1 p \)

\[\sum \omega_i(p) = 1\]
Interpolation of system coefficients

Nonlin. Interpolation of coefficients (system matrices):

\[
\dot{x} = \left( \sum_{i=1}^{s} \omega_i(p) A_i \right) x + \left( \sum_{i=1}^{s} \omega_i(p) b_i \right) u, \quad y = \left( \sum_{i=1}^{s} \omega_i(p) c_i^T \right) x
\]

\[
\sum_{i=1}^{s} \omega_i(p) = 1
\]

Traditional Reduction

Traditionally: apply one common projector pair \( V, W \):

\[
\dot{x} = \left( \sum_{i=1}^{s} \omega_i(p) A_i \right) \left( W^T V \right) x + \left( \sum_{i=1}^{s} \omega_i(p) b_i \right) u, \quad y = \left( \sum_{i=1}^{s} \omega_i(p) c_i^T \right) \left( W^T V \right) x
\]

Problem: \( V \) (and \( W \)) need many columns to well approximate all \( s \) local models! \( \rightarrow \) large reduced order.

(For instance, to match \( 2q \) moments of each local model, the reduced model’s order will be \( sq \), instead of \( q \) in non-parametric reduction)
New: Reduction by Local Projectors

Apply separate projectors $V_i$, $W_i$ to all local models:

$$
\dot{x} = \left( \sum_{i=1}^{s} \omega_i(p) A_i \right) x + \left( \sum_{i=1}^{s} \omega_i(p) b_i \right) u, \quad y = \left( \sum_{i=1}^{s} \omega_i(p) c_i^T \right) V_i \tag{\text{*}}
$$

+ Almost no additional numerical effort,
+ Much smaller reduced models (factor $s$ when matching same number of moments).

Open question: are we allowed to sum up physically different reduced vectors $\dot{x}$? Answer:

Not at once, but after giving the local reduced models a common physical interpretation of state variables (by applying state transformations $T_i$)

State Transformations $T_i$ (+ local projectors)

Define a linear combination $x^* = \mathbf{R} \cdot x$ of $q$ “important” state variables and transform all local reduced models, to represent these state variables:

$$
x_i^* = \mathbf{R} V_i x_{i,\text{red}} \quad \Rightarrow \quad x_i^* = T_i x_{i,\text{red}} \tag{\text{(*)}}
$$

$\Rightarrow$ In (\text{*}), substitute $V_i$ by $V_i T_i^{-1}$ and $W_i^T$ by $T_i W_i^T$, (with $T_i = \mathbf{R} V_i$).
How to choose $R$?

Option 1: by physical insight or from given output variables.

Option 2: so that the matrices $T_i = RV_i$ are well-conditioned or even $T_i = I$:

$$[I \cdots I] \approx R[V_i \cdots V_s] \Rightarrow R = [I \cdots I][V_i \cdots V_s]$$

Option 3: $R^T = V_{centre}$, (then, $T_{centre} = I$)

Option 4: $R^T = \text{svd}(V_1 \cdots V_s)$ or $R^T(p) = \text{svd}(\omega_1(p)V_1 \cdots \omega_s(p)V_s)$

Remark: Options 1 and 2 even work when original models are different size!

Summary: Interpolation of locally reduced models

Full order model:

$$\left( \sum_{i=1}^{s} \omega_i(p)E_i \right)x = \left( \sum_{i=1}^{s} \omega_i(p)A_i \right)x + \left( \sum_{i=1}^{s} \omega_i(p)b_i \right)u, \quad y = \left( \sum_{i=1}^{s} \omega_i(p)c_i^T \right)x$$

Reduced model using local projectors:

$$\left( \sum_{i=1}^{s} \omega_i(p)W_i^TE_iV_i \right)x = \left( \sum_{i=1}^{s} \omega_i(p)W_i^TA_iV_i \right)x + \left( \sum_{i=1}^{s} \omega_i(p)W_i^Tb_i \right)u, \quad (*)$$

$$y = \left( \sum_{i=1}^{s} \omega_i(p)c_i^TV_i \right)x$$

If required, substitute $V_i$ by $V_iT_i^{-1}$ and $W_i^T$ by $T_iW_i^T$ (where $T_i = RV_i$).
Types of Weighting Functions $\omega(p)$ / Interpolations

Matrices of the local reduced-order models

Explicit weights

- Linear interpolation
- Nonlinear interpolation

Implicit interpolation

- Spline interpolation
- Hermite interpolation
- RBF interpolation

$A_r = \omega_1(p) A^*_{r,1} + \omega_2(p) A^*_{r,2} + \cdots$

A_r

Part 2:
Moment Matching
for any value of $p$
Moments $m_j$

Transfer function:

$$G(s) = c^T (sE - A)^{-1} b = -c^T A^{-1} b - c^T A^{-1} E A^{-1} b s - ... - c^T (A^{-1} E)^j A^{-1} b s^j$$

Taylor series at $s_0 = 0$

$$\frac{1}{s} \left( \frac{1}{1 + \left( \frac{s}{s_0} \right)} \right)$$

Moments

Standard reduction matches moments only here, at $p_1$, $p_2$

Moment matching possible inbetween? Answer: yes

Krylov-Reduction, matching interpolated moments at any $p$

System:

$$E(p) \dot{x} = A(p) x + b(p) u, \quad y = c^T (p) x$$

Moments:

$$m_j(p_1) = c^T (A^{-1}_1 E)_j A^{-1}_1 b_1$$

$$m_j(p_2) = c^T (A^{-1}_2 E)_j A^{-1}_2 b_2$$

Interpolated moments:

$$m_j(p) = \omega_1(p) m_j(p_1) + \omega_2(p) m_j(p_2)$$

Reduction steps: 1) Find locally reduced models

$$W_i^T E V_i x = W_i^T A V_i x + W_i^T b_i u, \quad y = c^T V_i x$$

$$\left( W_i^T A V_i \right)^{-1} W_i^T E V_i \dot{x} = x + \left( W_i^T A V_i \right)^{-1} W_i^T b_i u, \quad y = c^T V_i x$$

where

$$V_i = [A^{-1}_1 b_1, (A^{-1}_1 E)_1, A^{-1}_1 b_1, ..., (A^{-1}_1 E)^{p-1}_1 A^{-1}_1 b_1]$$

$$W_i = [A^{-1}_1 c_1, (A^{-1}_1 E)_1 A^{-1}_1 c_1, ..., (A^{-1}_1 E)^{p-1}_1 A^{-1}_1 c_1]$$
Krylov-Reduction matching interpolated moments at any $p$

2) Add (weighted) reduced models up to the result:

$$E_r(p)\dot{x} = x + b_r(p)u, \quad y = c_r^T(p)x$$

where

$$E_r = \sum_{i=1}^{s_0} \omega_i (W_i^TAV_i)^{-1}W_i^TEV_i$$

$$b_r = \sum_{i=1}^{s_0} \omega_i (W_i^TAV_i)^{-1}W_i^Tb_i$$

$$c_r^T = \sum_{i=1}^{s_0} \omega_i c_i^TV_i$$

This parametric reduced model matches the first $q$ interpolated moments at any value of $p$!

---

Krylov-Reduction matching interpolated moments at any $p$

Proof (with $p \in [p_1, p_2]$):

$$m_{r_0} = c_r^Tb_r =$$

$$= [\omega_1 c_1^TV_1 + \omega_2 c_2^TV_2][\omega_1 (W_1^TAV_1)^{-1}W_1^Tb_1 + \omega_2 (W_2^TAV_2)^{-1}W_2^Tb_2]$$

$$= [\omega_1 c_1^TV_1 + \omega_2 c_2^TV_2][\omega_1 r_0 + \omega_2 r_0] = \omega_1 c_1^TA_r^{-1}b_1 + \omega_2 c_2^TA_r^{-1}b_2 = m_0$$

where we used $b_i = A_r A_i^{-1}b_i = A_r V r_0$ with $r_0 = e_1$

$$m_{r_1} = c_r^T E_r b_r = c_r^T E_r e_1$$

$$= \ldots = m_1$$

Remarks:

- Arnoldi can be used (instead of simple $V$, $W$ used above)
  requiring a transformation $T$ in low dimension, similar to part 1
  (see appendix).

- Other development points than $s_0=0$ can be used (Eid 2008).
The Beam Model

Parameter: Length $L$

Thickness and width: 10 mm
Young Modulus: $2 \times 10^5$ Pa.
Damping: Proportional/Rayleigh

Order of the original system: 720
Order of the reduced system: 5
4 local models; Weights: Lagrange Int. $R$: option 4; $s_0$: ICOP (Eid2009);

The Beam Model, order 24

Parameter: Length $L$

Order of the original system: 720
Order of the reduced system: 24
The Micro-Thruster Benchmark Model

Parameter: Film coefficient $k$
(from the convection boundary condition) varies between 1 and $10^9$

Order of the original system: 4725
Order of the reduced system: 7
20 local models employed for interpolation; $R$: option 4;
Weights: RBF-Based Interpolation with cubic basis functions.

Outlook I: Stability

System $\dot{x} = Ax$ is called $\gamma$-contractive, if $|x(t)| \leq e^{\gamma t} \cdot |x(0)|$
for any $x(0)$ and $t > 0$

Reduction by projection $\hat{x}_T = V_T^T A V x_T$ preserves $\gamma$-contractivity!

Idea: Make original model $\gamma$-contractive ($\gamma$ depending on the desired expansion point) by state transformation, and then reduce by projection.

The required state transformation can be found by a numerically cheap (mediocre) approximate solution of a Lyapunov-eq. (Castañé et al. 2009).
Outlook II: MOR of PCHD Models

Port Hamiltonian Systems

\[ x = (J - R)Qx + gu \]
\[ y = g^TQx \]

are stable, and passive with output

A new structure preserving reduction scheme:

The reduced model

\[ \dot{x} = (J_r - R_r)Q_r x + g_r u \]
\[ y = g_r^TQ_r x \]

with

\[ J_r = V^TQ_jQV \]
\[ R_r = V^TQRQV \]
\[ Q_r = (V^TQV)^{-1} \]
\[ g_r = V^Tg \]

and with \( V \) being a basis of the Krylov subspace

\[ K_q = \text{span}\{(J - R)Q - s_0 I)^{-1} g, ..., ((J - R)Q - s_0 I)^{-1} g\} \]

matches \( q \) moments around \( s=s_0 \) (Loh. et al 2009, Wolf et al 2009).

Outlook III

Nonlinear parametric reduction by interpolation of locally reduced linear models

Given

\[ \dot{x} = f(x, p) \]

Locally linear parametric representation (like in TPWL):

\[ \dot{x} = \sum_{i=1}^{s} \omega_i (x, p) (f_i + A_i (x - x_i)) \]
\[ f_i = f(x_i, p_i) \]
\[ A_i = \frac{\partial f_i}{\partial x} \bigg|_{x_i, p_i} \]

Reduced system:

\[ \dot{x}_i(t) = \sum_{i=1}^{s} \omega_i (V_i x_i(t), p) \cdot W^T (f_i + A_i (V_i x_i(t) - x_i)) \]

normalize \( \omega_i \) to have \( \sum \omega_i = 1 \)
Thank you for your attention.

References

5. A. C. Antoulas, Approximation of Large-Scale Dynamical Systems, SIAM, 2005.
Appendix

For numerical reasons, the projection matrices \( V_i, W_i \) are typically orthogonralized by the famous Arnoldi algorithm before use as projector. If we do so, vectors \( r_i \) that solve \( A_i^{-1}b = V_i r_i \) are no longer the same for any \( i \), and vectors \( r_i \) that solve \( A_i^{-1}E_i A_i^{-1}b_i = V_i r_i \) are no longer the same for any \( i \), which, however, was needed in the proof above. A remedy is the following:

- Calculate orthogonal projectors \( V_i^*, W_i^* \) using the Arnoldi algorithm as in conventional (non-parametric) model reduction. As a byproduct, the algorithm also delivers upper triangular non-singular matrices \( 3 \overline{H}_i \) and \( \overline{H}_i \) satisfying

\[
V_i = V_i^* H_i , \quad W_i = W_i^* H_i . \tag{37}
\]

- Out of them, choose one pair of matrices \( H_i \) and \( \overline{H}_i \), preferably belonging to a "central" or "average" value of the parameter or parameter set) and denote these two matrices by \( \overline{H}_i \) and \( \overline{H}_i \).

- For the reduction of all the local models, use the new projectors

\[
V_{new,i} = V_i^* H_i \overline{H}_i^{-1} , \quad W_{new,i} = W_i^* \overline{H}_i \overline{H}_i^{-1} . \tag{38}
\]

With this choice, all matrices \( V_i, W_i \) can be expressed from their substitutes \( V_{new, i}, W_{new, i} \) by

\[
V_i = A_i^{-1} b_i, A_i^{-1} E_i A_i^{-1} b_i, ... = V_i^* H_i \overline{H}_i^{-1} \overline{H}_i = V_{new, i} \overline{H}_i ,
\]

\[
W_i = A_i^{-1} c_i, A_i^{-1} E_i A_i^{-1} c_i, ... = W_i^* \overline{H}_i \overline{H}_i^{-1} \overline{H}_i = W_{new, i} \overline{H}_i .
\]

i.e. by multiplying the new projector with one common matrix, \( \overline{H}_i \) or \( \overline{H}_i \). The above proof of moment matching can now be repeated without essential changes.