On the stability of Krylov-based order reduction using invariance properties of the controllability subspace

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This article presents an algorithm to achieve stable reduced models using Krylov-based model order reduction for discrete time systems while matching a certain number of Markov parameters. By using the invariance properties of the controllability matrix suitable input and output Krylov subspaces are derived. The method is illustrated performing model reduction of four well-known benchmark problems.

1 Introduction

When the problem of control, design or simulation of large-scale systems is tackled, many difficulties that make the task often very difficult or even impossible, arise, mainly due to the high computational complexity of the involved algorithms which is commonly around $\mathcal{O}(n^3)$. This drawback generally limits the application of these approaches to systems of a few hundreds of states.

A possible solution to this problem is the reduction of the number of states of the system. To this aim, Krylov subspace methods are among the best choices nowadays, mainly because of their low computational complexity $\sim \mathcal{O}(n^2k)$ and low memory storage requirements $\sim \mathcal{O}(n^2k)$ (with $k$ the order of the reduced system). Briefly, the method consists on the projection of the large scale system into a particular subspace that can be chosen such that some properties of the original system are preserved. By now, these methods have also important drawbacks as no global bound for the approximation error exists and stability is not necessarily preserved. In this article, the problem of stability preservation in Krylov-based order reduction will be studied.

The literature in stability preserving Krylov-based model reduction can be divided in passivity preserving methods and stability preserving methods. As passive systems are a subgroup of stable systems passivity preserving methods preserve stability ([Fre00, OCP98]) for a given passive original system. The main drawback of these methods is that passivity is a property restricted to a narrow set of dynamical systems. An interesting group of passivity preserving methods are the interpolation based methods. In [Sor05], an interpolation based algorithm to preserve passivity is presented. The work is inspired by [Ant05b], where passivity is characterized through some interpolation conditions. The main result is that using interpolation at some spectral zeros of the original transfer function of the system, the reduced system preserves passivity and has the poles in the previous spectral zeros locations. This approach is theoretically very interesting, however, it has some practical drawbacks: the choice of some design parameters such as which spectral zeros are to be chosen or how to efficiently compute these spectral zeros, is not clear.

In the group of stability preserving methods, there are some works that preserve this property via post-processing ([BFF97, Gri97, JK97]). These works are mostly implemented for the SISO case and have a relatively high numerical effort compared to the classical case. In addition, they cannot preserve the moment matching after deleting the unstable poles, and can not always
guarantee to find a stable reduced model with a finite number of restarts. On the other side, in [GA06, You06, Ant07] Krylov based approximation methods that combine guaranteed stability and moment matching are presented. However, these approaches are computationally too expensive due to the computation of Lyapunov equations or, in the case of the last work, they are not applicable due to important numerical problems.

In this article, invariance properties of the controllability subspace will be used to design a Krylov-based model order reduction method which preserves stability of the reduced models.

2 Preliminaries

In this section the basic aspects of the formulation that will be used in the article will be explained. Consider a linear time invariant dynamical in state space representation:

\[
\dot{x}(t) = Fx(t) + Gu(t),
\]

\[
y(t) = Hx(t) + Ju(t),
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}\) the input of the system, \(y(t) \in \mathbb{R}\) the output, \(F \in \mathbb{R}^{n \times n}\) the state matrix, \(g \in \mathbb{R}^n\) the input matrix, \(h^T \in \mathbb{R}^n\) is the output-state matrix and \(J \in \mathbb{R}\) the output matrix. For the ease of presentation, only the single-input single-output (SISO) case will be tackled. However, the results can be easily extended to the multi-input multi-output case (MIMO).

The SISO dynamical system presented in Eq. (1) has the following transfer function:

\[
G(s) = h(sI - F)^{-1}g + J \equiv \begin{bmatrix} F & g \\ h & J \end{bmatrix},
\]

where \(F \in \mathbb{R}^{N \times N}, g, h^T \in \mathbb{R}^N\) and \(J \in \mathbb{R}\).

At this point, the system is discretized applying a bilinear transformation which corresponds to applying a shift \(\mu\) from the left and right to the \(A\) matrix. Then, the following system is obtained and the transformation equalities are the following:

\[
H(z) \equiv \begin{bmatrix} A & b \\ c & d \end{bmatrix} = \begin{bmatrix} (\mu I + F)(\mu I - F)^{-1} & \sqrt{2\mu}(\mu I - F)^{-1}g \\ \sqrt{2\mu c(\mu I - F)^{-1}} & J + h(\mu I - F)^{-1}g \end{bmatrix},
\]

where \(A \in \mathbb{R}^{N \times N}, b, c^T \in \mathbb{R}^N\) and \(d \in \mathbb{R}\). The equivalence between the continuous and the discrete time expressions is \(G(s) = H\left(\frac{\mu}{\pi}z\right)\). It is straightforward to see that this transformation is equivalent to the discretization of the system using a sampling frequency \(f = \frac{\mu}{2\pi}\). In the next sections, it will be always assumed that the original systems are in the discrete time formulation.

2.1 Stability and passivity

As the topic of stability preservation is studied in this article, some basic concepts such as stability and passivity have to be revisited.

A system is stable if and only if there exists a positive definite matrix \(\mathcal{P}\) such that the following Lyapunov equation is satisfied:
\[
F^T P + PF < 0. 
\]

Eq. (5) can be explicitly solved. Indeed, we can pick any \( Q = Q^T > 0 \) and then solve the linear equation \( F^T P + PF = -Q \) for the matrix \( P \), which is guaranteed to be positive definite if the system is stable.

Note that when the bilinear transformation is applied to the reformulated Eq. (5) the Stein Equation is obtained:

\[
P = A PA^T + \tilde{Q},
\]

where \( A \) is the discrete state matrix defined in Eq. (4). Therefore, a discrete time system is stable if the following equation is feasible:

\[
\begin{align*}
A PA^T - P + \tilde{Q} &= 0; \\
P &> 0, \\
\tilde{Q} &> 0.
\end{align*}
\]

A stronger property than stability is passivity. A system is passive if it does not create energy, i.e. \( \int_0^t u(t)^T y(t) dt \geq 0 \), where \( u(t) \) is the input and \( y(t) \) is the output of the system. Passivity can be also expressed in terms of the transfer function \( G(s) \) being positive-real, which means that \( G(s) + G(s)^* \geq 0 \), for all \( \text{Re}(s) > 0 \). But also, if the system is passive the following LMI is feasible:

\[
\begin{bmatrix}
F^T P + PF & P g - h \\
g^T P - h & -J^T - J
\end{bmatrix} \leq 0, \\
P > 0.
\]

Note that passivity implies stability but not viceversa.

### 2.2 Krylov-based model reduction

At this point, some well known properties of the linear systems that will be important in the development of the paper together with basic concepts of Krylov-based order reduction, will be presented.

A discrete-time linear SISO system can be viewed as an operator between two spaces \( \mathbb{U} \) and \( \mathbb{Y} \) and its impulse response (assuming \( d = 0 \)) can be represented as \( y = \begin{bmatrix} cb & cAb & cA^2b & \cdots \end{bmatrix} \).

This values can be represented in an infinite-dimensional Hankel matrix as seen in the following equation:

\[
\mathcal{H} = \begin{bmatrix}
\eta_1 & \eta_2 & \eta_3 & \cdots \\
\eta_2 & \eta_3 & \eta_4 & \cdots \\
\eta_3 & \eta_4 & \eta_5 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad \eta_i = cA^{i-1}b, \tag{8}
\]

where \( \mathcal{H} \) is a Hankel matrix and \( \eta_i \) are the so called Markov parameters. At the same time, note that \( \mathcal{H} \) can be build as a product of the infinite reachability \( R \) and observability \( O \) matrices: \( \mathcal{H} = OR \), which are defined as follows:
\[ \mathcal{R} = \begin{bmatrix} b & Ab & A^2b & \cdots \end{bmatrix}, \quad (9) \]

\[ \mathcal{O} = \begin{bmatrix} c^T & A^Tc^T & (A^T)^2c^T & \cdots \end{bmatrix}^T. \quad (10) \]

In Krylov-based order reduction methods for discrete time systems, the classical goal is to reduce the system using a projection such that the original and the reduced system have a number of Markov parameters in common. The projection matrices that will be used for the reduction are called \( \mathbf{V} \in \mathbb{R}^{n \times r} \) and \( \mathbf{W}^T \in \mathbb{R}^{r \times n} \). \( \mathbf{V} \) will be used to project the state vector according to \( \mathbf{V} \hat{x} = x \). This projection introduces a residual \( \mathcal{E} \) that can be set to zero by projection onto the subspace spanned by \( \mathbf{W}^T \). Hence, by considering the condition \( \mathbf{W}^T \mathcal{E} = 0 \) the reduced state space model is defined as follows:

\[
\begin{align*}
\mathbf{W}^T \mathbf{V} \hat{x}_{k+1} &= \mathbf{W}^T \mathbf{A} \hat{x}_k + \mathbf{W}^T \mathbf{b} \mathbf{u}, \\
y &= c \mathbf{V} \hat{x}_k + d \mathbf{u}.
\end{align*}
\]

Therefore, it can be expressed by the transfer function and the system matrix formulation indicated in the next equation:

\[
H_r(s) = c_r (sI_r - A_r)^{-1} c_r + d_r = \begin{bmatrix} (W^T \mathbf{V})^{-1}W^T \mathbf{A}V \\ c^V \\ d \end{bmatrix}, \quad (12)
\]

where \( \mathbf{W}, \mathbf{V} \in \mathbb{R}^{n \times r} \). Now the question is how to choose the projection matrices so that a number of Markov parameters are matched. Here, the observability and reachability subspaces defined in Eqs. (9) and (10) will play and important role.

**Theorem 1** \([\text{Ant05a}]\) Let the projection matrix \( \mathbf{V} \) be the first \( r \) columns of \( \mathcal{R} \) and \( \mathbf{W} \) be any left inverse of \( \mathbf{V} \). Then, \( r \) Markov parameters are matched between the original and the reduced system.

**Theorem 2** \([\text{Ant05a}]\) When \( r \) columns of the matrices \( \mathcal{R} \) and \( \mathcal{O} \) are taken as projection matrices \( \mathbf{V} \) and \( \mathbf{W}^T \) respectively, then \( 2r \) Markov parameters (\( \eta_i \)) of the original and the reduced system are matched.

However, when the option presented in Theorem 2 is chosen, the reduced model is *unique and not necessarily stable*. For this reason, a valid idea is to use the input Krylov subspace \( \mathbf{V} \) to match a number of Markov parameters and use \( \mathbf{W}^T \) to preserve stability.

### 3 Main results

In order to preserve stability of the original system in the reduction process, a result from subspace identification (\([\text{Cho94, Mac95}]\)) where a stable \( \mathbf{A} \) matrix is identified will be used. Let us define the reduced controllability matrices of column rank \( r \) and \( r - 1 \) and the shifted invariant \( \hat{\Gamma}_0 \) of the second one as follows:
\[ \hat{\Gamma} = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{r-2}b & A^{r-1}b \end{bmatrix}, \]  
(13) 
\[ \hat{\Gamma}_0 = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{r-2}b & 0 \end{bmatrix}, \]  
(14) 
\[ \hat{\Gamma}_0^{-} = \begin{bmatrix} A_{r-3} & A_{r-2} & A_{r-1} & \cdots & A^0 \end{bmatrix}. \]  
(15)

with \( \hat{\Gamma}^T = QR \) where \( Q^TQ = I \) and \( R \) is upper-triangular. In the same way, the shifted invariant of a tall matrix such as \( \Phi = \hat{\Gamma}^T \) is defined as \( \Phi^T \).

Defining the Moore-Penrose pseudoinverse as \(^+\), the following equivalences hold: \( \hat{\Gamma}_0^T = Q_0 R, \) \( \hat{\Gamma}_0^{-} = Q_0 R \) and \( \hat{\Gamma}_0^{+,T} = R^+ Q^T \) and the following Theorem can be stated:

**Theorem 3** Given a stable discrete time state space model, stable reduced model can be obtained using the following projection matrices:

\[
\begin{aligned}
V &= \hat{\Gamma}_0, \\
W^T &= \hat{\Gamma}^+.
\end{aligned}
\]  
(16)

The state matrix of the reduced system is \( A_r = \hat{\Gamma}^{+,T} \hat{\Gamma}_0^{-} \) and the first \( r - 1 \) Markov parameters of the original and the reduced system match.

**Proof:**
>From (12) it is known that \( A_r = (W^T V)^+ W^T A V \) can be defined as the reduced state matrix. Choosing \( V = \hat{\Gamma}_0 \) and \( W^T = \hat{\Gamma}^+ \) leads to:

\[
A_r = (\hat{\Gamma}^{+,T} \hat{\Gamma}_0)^{-1} \hat{\Gamma}^{+,T} A \hat{\Gamma}_0,
\]

\[
= \hat{\Gamma}^{+,T} \hat{\Gamma}_0^{-} = \begin{bmatrix} Q_{r-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},
\]

where \( Q_{r-1} \) has full rank. Therefore \( A_r \) has an eigenvalue in \( 0 \) which eigenvector is \( \begin{bmatrix} \mathbf{0}_{r-1} & 1 \end{bmatrix}^T \).

Now, the stability of the rest of the subspace spanned by \( Q_{r-1} \) has to be proved. If \( \omega \) is an eigenvector of \( Q_{r-1} \) such that \( A_r \omega = \lambda \omega \), it follows that,

\[
\lambda Q_0^{+,T} \omega = \begin{bmatrix} Q_0^{+,T} A_r \omega \\ Q_0^{+,T} \hat{\Gamma}_0^{-} \omega \\ Q_0^{+,T} QR^+ R^+ Q_0^{+,T} \omega \end{bmatrix}.
\]

Let \( \| \cdot \| \) be the Euclidean norm for vectors and the compatible induced norm for matrices. Using the *Cauchy Schwarz inequality* the following equations are obtained:

\[
|\lambda| \times \| Q_0^{+,T} \omega \| = \| Q_0^{+,T} QR^+ R^+ Q_0^{+,T} \omega \| \leq \| Q_0^{+,T} Q \| \times \| Q_0^{+,T} \omega \|.
\]
Now considering \( P = (Q_0^T Q)^T (Q_0^T Q), \) \( P^2 = P \) and at the same time \( P^T = P \). Therefore, \( P \) is an orthogonal projector [GL83] and all its eigenvalues take the values 0 or 1. Furthermore, it is known that the singular values of \( (Q_0^T Q)^T (Q_0^T Q) \) are the non-negative square roots of the eigenvalues of \( (Q_0^T Q)^T (Q_0^T Q) \). It is known that \( \|Q_0^T \omega\| \) is given by its largest singular value, leading to

\[
|\lambda| \times \|Q_0^{1:T} \omega\| \leq \|Q_0^{1:T} \omega\|
\]

and \( |\lambda(E^*_r A_r)| \leq 1 \). □

**Remark 1** It has been shown that matching the Markov parameters in discrete time is equivalent to matching the moments about \( s_0 = \mu \). Accordingly, guaranteeing stability in the discrete case corresponds to guaranteeing stability of the continuous time reduced model obtained by matching the moments about \( s_0 = \mu \).

### 3.1 Numerical and computational aspects

This method allow us to achieve stable reduced models that match the Markov parameters. However, the computation of the controllability subspace is often ill-conditioned. When this is the case, the reduced controllability matrix can be obtained taking the first \( r \) columns of the Cholesky factor of the discrete controllability gramian, solution of \( A P A^T - P + BB^T = 0 \). This fact can be considered a drawback of the presented method as the computational complexity for solving a Stein equation is \( O(n^3) \).

Regarding the computation complexity of pseudoinverses, in [CKP07], the solution of \( n \times r \) matrices is treated. The complexity of those protocols is \( r^4 + n^2 r \). Our main point is, however, that the advantage can be substantial in case \( r \) is much smaller than \( n \). For example, if only \( r = \sqrt{n} \) is accomplished, the complexity goes down from \( r^5 \) to \( r^{2.5} \) already. For large scale systems a typical case would be to reduce a model of 100,000—500,000 states down to a 50—100 states. Therefore, it can be always assume that \( r \ll n \) and the computational complexity of the pseudoinverse decreases rapidly to around \( r^2 \).

### 3.2 Algorithm

An algorithm for obtaining stable reduced models is presented in the following lines. Note that the solution of the Stein equation is only needed in the case that the reduced controllability subspace is ill-conditioned.

**Input:** Discrete time system \( A, b, c, d \). Order of the reduced system \( r \).

**Output:** reduced stable system \( A_r, b_r, c_r, d_r \)

\[
\mathbf{R} = \text{REDUCED CONTROLLABILITY SUBSPACE}(\mathbf{A}, \mathbf{b}, r);
\]

if \( \text{rank}(\mathbf{R}) \ll r \) then

\[
\mathbf{P} = \text{SOLVE STEIN}(\mathbf{A}, \mathbf{B});
\]

\[
\mathbf{R} = \text{CHOLESKY}(\mathbf{P});
\]

\[
\mathbf{R} = \mathbf{R}[1 : r, :];
\]

end

\[
[\mathbf{V}, \mathbf{W}] = \text{COMPUTE KRYLOV SUBSPACES}(\mathbf{R});
\]

\[
[\mathbf{A}_r, \mathbf{b}_r, \mathbf{c}_r, \mathbf{d}_r] = \text{PROJECTION}(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{V}, \mathbf{W});
\]

**Algorithm 1:** Pseudocode of the algorithm
Table 1: Simulation time and results for the algorithm

<table>
<thead>
<tr>
<th>System Information and Objectives</th>
<th>Algorithm</th>
<th>Classical Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>System</strong></td>
<td><strong>T_s</strong></td>
<td><strong>O</strong></td>
</tr>
<tr>
<td>Build</td>
<td>48</td>
<td>0.01</td>
</tr>
<tr>
<td>• 48</td>
<td>0.001</td>
<td>12</td>
</tr>
<tr>
<td>• 48</td>
<td>0.00001</td>
<td>4</td>
</tr>
<tr>
<td>Beam</td>
<td>348</td>
<td>0.05</td>
</tr>
<tr>
<td>Fom</td>
<td>1006</td>
<td>0.05</td>
</tr>
<tr>
<td>Tub</td>
<td>63</td>
<td>0.002</td>
</tr>
</tbody>
</table>

**Order**: Order for which the reduced controllability subspace becomes ill-conditioned.

†: The reduced controllability subspace is ill-conditioned. Therefore, the solution of the Stein equation is needed.

The procedure is the following: first, the reduced controllability subspace is calculated. Then, if the rank of the controllability matrix does not have a value close to the desired reduced order, the Cholesky decomposition of the solution of the Stein equation $AQA^T - Q + BB^T = 0$ has to be calculated and the first $r$ columns should be taken. In the next step, the projection matrices are defined as in Eq. (16), and finally, the projection is performed according to Eq. (12).

In the next section, this algorithm is applied to perform the model reduction of four well-known benchmark problems.

4 Examples

Three benchmarks problems presented in [CVD02] have been used to illustrate the presented approach. The build system is a model of a building (the Los Angeles University Hospital) with 8 floors each having 3 degrees of freedom, namely, displacements in $x$ and $y$ directions and rotation. Secondly, the clamped beam model (beam) is considered. It has 348 states and is obtained by spatial discretization of an appropriate partial differential equation. The third example is the benchmark problem specified by fom which is a dynamical system of order 1006. Finally, the new algorithm is applied to reduce the tub [SQP07] which is the model of an acoustic tube model for active noise control purposes.

In Table 4 the results of the algorithm for all the benchmark problems are presented. Note that the table is divided in three parts. In the first part, the basic characteristics of the system are presented as well as the order to which it will be reduced and the order for which the reduced controllability subspace becomes ill-conditioned. Remark that, in this case, the Krylov subspace has to be calculated as the Cholesky decomposition of the controllability gramian and, therefore, the computational cost is higher. In the second part of the Table, the time results for the reductions are presented together with the time of solving one Stein equation. Finally, the stability results for the classical one-sided method are illustrated in the last column.

In Fig. 1 the frequency response of the most complex models (the tub and the build models) together with the response of the reduced systems are presented. Note that different orders for the reduced system have been chosen in order that the approximation results can be compared. As a result, good fitting for all the presented benchmark problems have been achieved. For example, the build system has been reduced from order 48 to order 12 with excellent results in the desired bandwidth. This projection lead always to unstable systems when the classical
methods were used. This is also the case when reducing the tube model.
The main problem of the algorithm is that the computation of $r$ consecutive and linearly independent vectors of the controllability subspace is not always guaranteed and in the worst case scenario it has to be obtained numerically or via the Cholesky decomposition of the controllability gramian.

5 Conclusions

A method to obtain stable reduced models using Krylov-based model order reduction has been presented. This method, uses invariance properties of the controllability subspace to preserve stability. The main advantage of the procedure is that no condition on the original system matrices is imposed and stability of the reduced models is always guaranteed. Simulation results for four benchmark problems illustrate the performance of the suggested method and excellent results are obtained. Even though numerical problems may occur, it has been shown that they can be completely solved via the Cholesky decomposition of the solution of one single Stein equation.

References


